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GLOBAL SOLUTIONS FOR AN AGE-DEPENDENT MODEL OF NUCLEATION, GROWTH AND AGEING WITH HYSTERESIS

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Abstract. In this article we establish the global well-posedness of a recent model proposed by Noguera, Fritz, Clément and Baronnet for simultaneously describing the process of nucleation, growth and ageing of particles in thermodynamically closed, initially supersaturated systems. This model, which applies to precipitation in solution, vapor condensation and crystallization from a simple melt, can be seen as a highly nonlinear age-dependent population problem involving a delayed birth process and a hysteresis damage operator.

1. **Introduction.** In a recent work [4], Noguera, Fritz, Clément and Baronnet proposed a unified mathematical model that accounts for the process of nucleation, size-dependent growth, dissolution and ripening of particles (or droplets) in a thermodynamically closed, initially supersaturated system. The model is relevant for precipitation in solution, vapor condensation as well as crystallization from a simple melt, and compared to popular population balance models, it explicitly keeps track of the time evolution of any particle nucleated in the system. Given three independent dimensionless parameters u, w and J, the master equations read

$$\begin{aligned} n(s,t) &= \frac{2u}{\ln^3 S(s)} + 3w\int_s^t n^{2/3}(s,t')\Big(S(t') - \exp\Big(\frac{2u}{n(s,t')}\Big)^{1/3}\Big)\,\mathrm{d}t' \\ S(t) &= S_0 - J\int_0^t \exp\Big(-\frac{u}{\ln^2 S(s)}\Big)\big(n(s,t)-1\big)\,\mathrm{d}s \end{aligned} \qquad (1) \quad \{\texttt{orsys}\}$$

where $n(s,t)$ and $S(t)$ represent at each time t the size of every particle nucleated at a previous time $s \le t$ and a saturation index measuring the deviation from thermodynamic equilibrium (corresponding to $S = 1$), respectively. More specifically, as the size n is measured in terms of *growth units* – i.e., elementary units constituting the particles – it only makes sense for values larger or equal to 1, which is implicitly assumed in [4]. In order to take this feature into account we therefore had to add an *hysteresis operator* to the original equations and consider a modified system, as will soon be detailed – see System (6) below.

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1.1. **An atypical age-dependent problem.** In the sequel we shall consider that s and t belong to triangles of the form

$$D_T := \{(s,t) : 0 \le s \le t < T\} \qquad \text{with} \quad T \le \infty,$$

and rewrite the above system as

{newsys}

$$\begin{aligned} &\partial_t \rho(s,t) = w\Big(S(t) - \exp\big(\frac{\nu}{\rho(s,t)}\big)\Big) \qquad \text{for } t \ge s \\ &\rho(s,s) = \frac{\nu}{\ln S(s)} \\ &S(t) = S_0 - J \int_0^t \exp\Big(-\frac{\nu \rho^2(s,s)}{2}\Big)\big(\rho^3(s,t) - 1\big)\,\mathrm{d}s \end{aligned} \tag{2}$$

with $\rho := n^{1/3}$ and $\nu := (2u)^{1/3}$. Clearly, it is possible to remove the unknown function S: although this does not simplify the equations we observe that by introducing the *age* variable $a := t - s \in [0,t]$ and setting $\ell(a,t) := \rho(t-a,t)$, this allows to interpret the above system as an atypical age-dependent population problem involving individuals born at positive times, i.e.,

{adp}

$$\left.\begin{aligned} \big(\partial_t + \partial_a\big)\ell(a,t) &= G(\ell(\cdot,t), a) \\ \ell(0,t) &= F(\ell,t) \end{aligned}\right. \qquad \text{for } 0 \le a \le t, \tag{3}$$

with balance and (delayed) birth laws respectively given by

$$G(\phi, a) := w\Big(\exp\big(\nu/\phi(0)\big) - \exp\big(\nu/\phi(a)\big)\Big),$$

(for any continous ϕ) and

{birth}

$$F(\ell,t) := \nu\left[\ln\Big(S_0 - J\int_0^t \exp\Big(-\frac{\nu \ell^2(0,t-a)}{2}\Big)\big(\ell^3(a,t) - 1\big)\,\mathrm{d}a\Big)\right]^{-1} \tag{4}$$

(see [9] for an introduction to such equations, and [6] for a comprehensive mathematical treatment of biological scenarios, involving structured population dynamics). Note that problems involving delayed birth process have already been studied by several authors, see e.g. [3, 7, 1], but none with such laws, at least to our knowledge.

1.2. **Modeling with hysteresis.** As already pointed out, we shall not consider the system (2) as it is, but add an hysteresis parameter to it. Indeed, because the function $n = \rho^3$ actually represents the particles volume in terms of *growth units*, it only makes sense for values larger or equal to 1. Now, although such a property is implicitly assumed in [4], it has no reason to be fulfilled by the solutions of (2). Therefore we have introduced the auxiliary parameter

{chi}

$$\chi(s,t) := H\Big(\inf_{t' \in [s,t]} \rho(s,t') - 1\Big) \qquad \text{with} \quad H(x) := \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

and considered the following system:

{sys}

$$\begin{aligned} &\partial_t \rho(s,t) = w\chi(s,t)\Big(S(t) - \exp\big(\frac{\nu}{\rho(s,t)}\big)\Big) \qquad \text{for } t \ge s, \\ &\rho(s,s) = \frac{\nu}{\ln S(s)} \\ &S(t) = S_0 - J \int_0^t \exp\Big(-\frac{\nu \rho^2(s,s)}{2}\Big)\big(\rho^3(s,t) - 1\big)\,\mathrm{d}s. \end{aligned} \tag{6}$$

Here $\chi(s,t)$ represents the *activity* at time t of the class $\rho(s,\cdot)$: it is 1 as long as the class has more than one growth unit, and jumps from 1 to 0 as $\rho(s,\cdot)$ reaches the critical value 1. Then the class is like "dead": its size $\rho(s,\cdot)$ remains equal to 1, and according to the nucleation law (4), it does not contribute to the dynamics anymore.

Note that χ indeed models an hysteresis phenomenon in the classical sense: the operator $\Lambda : \mathcal{C}([0,T];\mathbb{R}) \to \mathcal{C}([0,T];\mathbb{R})$ defined by

$$\Lambda\phi(t) := H\Big(\inf_{t'\in[s,t]} \phi(t') - 1\Big)$$

is an hysteresis operator of damage type – according to the terminology of [8] – and we have $\chi(s,t) = [\Lambda\rho(s,\cdot)](t)$. Again, let us mention that well-posedness of integro-differential problems involving hysteresis operators is already a substantial area of research (see e.g. the works of Darwish, Krejčí, Minchev, Sprekels, ...), but as far as we know, there is no available result in the context of age structured population dynamics.

Let us end this introduction by stating necessary bounds for S: first, we easily infer from $\rho \geq 1$ that the initial data S_0 is an upper bound for S, hence the natural assumption $S_0 < e^\nu$ suffices to ensure that every new class nucleates with a size $\rho(s,s)$ greater than 1. Second, we observe that 1 is a lower bound for S, in that the solution (more precisely, the size of each new born particles) blows up when that value is reached. Note that this is not contradictory with the fact that $S = 1$ corresponds to the thermodynamic equilibrium, because the nucleation process occurs at an *instantaneous rate* $F := F_0 \exp(-u/\ln^2 S)$ which rapidly tends to 0 as S tend to 1.

Although continuity is a natural requirement for S and $\rho(s,\cdot)$, it is not clear whether ρ must be continuous with respect to its first argument. We shall nevertheless look for solutions (ρ, S) to (6) in spaces of the form $\mathcal{R}_T \times \mathcal{S}_T$, which are defined for any given final time T and initial value S_0 by

$$\mathcal{R}_T := \{\rho \in \mathcal{C}(D_T;\mathbb{R}) : 1 \leq \rho(s,t)\}, \quad \mathcal{S}_T := \{S \in \mathcal{C}([0,T];\mathbb{R}) : 1 < S(t) \leq S_0 < e^\nu\}.$$

Our main result is then the following.

Main Theorem. *For any set of positive dimensionless parameters $\nu = (2u)^{1/3}$, w, J, and initial data $S_0 \in (1, e^\nu)$, the system (6) possess a unique global solution (ρ, S) on $\mathcal{R}_\infty \times \mathcal{S}_\infty$. Moreover the solution is locally stable with respect to the parameters, in the sense that if (ρ, S) and $(\tilde\rho, \tilde S)$ are the solutions corresponding to admissible parameters $D := (\nu, w, J, S_0)$ and $\tilde D := (\tilde\nu, \tilde w, \tilde J, \tilde S_0)$, respectively, then for any $T < \infty$ we have*

$$\|S - \tilde S\|_{L^\infty(0,T)} \leq C\|D - \tilde D\|_{\ell^\infty}$$

for a constant C that only depends on T and on $M := \|(D, \tilde D)\|_{\ell^\infty}$.

Note that no stability result is expected with respect to ρ, due to the fact that this variable may grow arbitrarily as S approaches 1. The proof is organized as follows. In Section 2 we start by considering finite times $T < \infty$ and establish existence and uniqueness of solutions to an auxiliary problem: Given $S \in \mathcal{S}_T$, find

$\rho_S \in \mathcal{R}_T$ such that

$$\begin{aligned} &\partial_t \rho_S(s,t) = w\chi(s,t)\Big(S(t) - \exp\big(\frac{\nu}{\rho_S(s,t)}\big)\Big) \qquad \text{for } t \geq s, \\ &\rho_S(s,s) = \frac{\nu}{\ln S(s)} \end{aligned} \tag{7}$$

{parsys}

holds. Next we study solutions to the complete system in Section 3: by showing that the mapping $Q_T : \mathcal{S}_T \to \mathcal{C}([0,T);\mathbb{R})$, defined for any $T < \infty$ by

{QT}

$$Q_T S(t) := S_0 - J\int_0^t \exp\Big(-\frac{\nu\rho_S^2(s,s)}{2}\Big)\big(\rho_S^3(s,t) - 1\big)\,\mathrm{d}s, \tag{8}$$

where ρ_S is the unique solution of (7), possess unique fixed points $S = S_T$ for sufficiently small T, we first establish the existence of a maximal solution to (6). We finally show that the solution is always global, and in Section 4 we establish the local stability.

2. Well-posedness of the partial problem involving hysteresis.

In this section we shall prove existence and uniqueness of solutions to Problem (7), and for that purpose we define a mapping $P_{S,T} : \mathcal{R}_T \to L^\infty(D_T)$ for $T < \infty$ and $S \in \mathcal{S}_T$, as follows. For any $\rho \in \mathcal{R}_T$, $s \in [0,T)$, we first denote by $f_{S,\rho} \in \mathcal{C}(D_T;\mathbb{R})$ the function

$$f_{S,\rho}(s,t) := \frac{\nu}{\ln S(s)} + w\int_s^t \Big(S(t') - \exp\big(\frac{\nu}{\rho_S(s,t')}\big)\Big)\,\mathrm{d}t'$$

and let

$$\tau(S,\rho,s) := \inf\{t \in [s,T) : f_{S,\rho}(s,t) \leq 1\}$$

be the first time where $f_{S,\rho}(s,\cdot)$ reaches 1 (remember that $S(s) \leq S_0 < e^\nu$, hence $f_{S,\rho}(s,s) > 1$, for any $S \in \mathcal{S}_T$). For later purposes we shall adopt the convention that $\tau(S,\rho,s) := T$ whenever $f_{S,\rho}(s,\cdot)$ is greater than 1 over $[s,T)$, hence $\tau(S,\rho,s)$ is always in $[s,T]$, and $\tau(S,\rho,s) < T$ implies $f_{S,\rho}(s,\tau(S,\rho,s)) = 1$. Next we set for any $(s,t) \in D_T$

{PST}

$$(P_{S,T}\rho)(s,t) := \begin{cases} f_{S,\rho}(s,t) & \text{if} \quad s \leq t \leq \tau(S,\rho,s) \\ 1 & \text{if} \quad \tau(S,\rho,s) < t < T. \end{cases} \tag{9}$$

As it can be checked, ρ is a fixed point of $P_{S,T}$ if and only if it is a solution to (7); hence we can derive the well-posedness of the latter by applying a fixed point theorem to $P_{S,T}$. Let us recall that according to the Picard-Banach theorem (see e.g. [2]), if a mapping $m : E \to E$ is contracting on the Banach space E, then it possess a unique fixed point. For later purposes we also recall the following elementary corollary: if (at least) one iterate m^p is contracting on E, then again m possess a unique fixed point.

Now, it is clear that the function $P_{S,T}\rho$ is always bounded below by 1, and it is easily seen that it is Lipschitz, hence continuous, with respect to its second argument t: indeed we have

{Prho lip}

$$\big|\partial_t(P_{S,T}\rho)(s,t)\big| \leq w\Big|S(t) - \exp\Big(\frac{\nu}{\rho(s,t)}\Big)\Big| \leq w(e^\nu - 1), \tag{10}$$

but for general S and ρ, it has no reason to be continuous with respect to s. One can think for instance of the case where $f_{S,\rho}$ is lower than 1 on a ball $B_2\big(m,r\big) := \{(s,t) \in D_T : \|(s,t) - m\|_2 \leq r\}$ and greater than 1 outside $B_2\big(m,R\big)$ with $R > r$, which is always possible if S is bounded away from e^ν (and for sufficiently large

T), by carefully choosing $\rho \geq 1$. In order to apply a fixed point theorem we shall therefore restrict ourselves to the subset

$$\mathcal{R}_{S,T} := \{\rho \in \mathcal{R}_T : \rho \leq P_{S,T}\rho\} \subset \mathcal{R}_T,$$

which, apart from clearly containing all the fixed points of $P_{S,T}$, enjoys the following properties.

Lemma 2.1. *For any $S \in \mathcal{S}_T$, $\mathcal{R}_{S,T}$ is a closed, non-empty subset of $\mathcal{R}_T$ such that*

$$P_{S,T}(\mathcal{R}_{S,T}) \subset \mathcal{R}_{S,T}, \tag{11}$$ {submap}

moreover the mapping $P_{S,T}$ is contracting on $\mathcal{R}_{S,T}$ when equipped with the norm

$$\|\rho\|_{\infty,k} := \sup_{(s,t)\in D_T} e^{-k(t-s)}|\rho(s,t)|, \qquad k := w\nu e^{\nu}, \tag{12}$$ {knorm}

which clearly satisfies $e^{-kT}\|\rho\|_{L^\infty(D_T)} \leq \|\rho\|_{\infty,k} \leq \|\rho\|_{L^\infty(D_T)}$.

Before going further, we already observe that these properties permit to apply the Picard-Banach fixed point theorem, and by using (10), establish the following

Theorem 2.2. *For any $S \in \mathcal{S}_T$, the mapping $P_{S,T}$ possess a unique fixed point ρ_S in $\mathcal{R}_T$. In particular, the auxiliary problem (7) involving hysteresis admits a unique (continuous) solution. Moreover this solution is Lipschitz with respect to its second argument:*

$$\|\partial_t \rho_S\|_{L^\infty(D_T)} \leq w(e^{\nu} - 1). \tag{13}$$ {rho lip}

In order to prove Lemma 2.1 we shall begin with an elementary property of the mapping $P_{S,T}$.

Lemma 2.3. *For any $S \in \mathcal{S}_T$, we have $P_{S,T}\rho_1 \leq P_{S,T}\rho_2$ as long as $\rho_1 \leq \rho_2$.*

Proof. Take ρ_1 and ρ_2 in $\mathcal{R}_T$, such that $\rho_1 \leq \rho_2$ on D_T: since $x \to -\exp(\nu/x)$ is increasing on $[1,\infty)$ we have $f_{S,\rho_1}(s,t) \leq f_{S,\rho_2}(s,t)$ for any $(s,t) \in D_T$, hence $\tau(S,\rho_1,s) \leq \tau(S,\rho_2,s)$ by construction. In particular, for any $s \in [0,T)$ we have

$$P_{S,T}\rho_2(s,t) = 1 = P_{S,T}\rho_1(s,t) \qquad \text{for} \qquad \tau(S,\rho_2,s) \leq t < T,$$

$$P_{S,T}\rho_2(s,t) > 1 = P_{S,T}\rho_1(s,t) \qquad \text{for} \qquad \tau(S,\rho_1,s) \leq t < \tau(S,\rho_2,s),$$

and finally for $s \leq t < \tau(S,\rho_1,s)$, we have

$$P_{S,T}\rho_2(s,t) - P_{S,T}\rho_1(s,t) = -w\int_s^t \Big(\exp\big(\frac{\nu}{\rho_2(s,t')}\big) - \exp\big(\frac{\nu}{\rho_1(s,t')}\big)\Big)\,\mathrm{d}t' \geq 0,$$

which ends the proof. $\square$

Let us now address the important properties of the set $\mathcal{R}_{S,T}$.

Proof of Lemma 2.1. First of all we observe that by construction of $P_{S,T}$, $\mathcal{R}_{S,T}$ contains the constant function $\rho(s,t) = 1$, hence it is a *non empty* subset of $\mathcal{R}_T$. In order to show that it is *closed* (with respect to the sup norm), let us consider a sequence $\rho_n \in \mathcal{R}_{S,T}$, $n \in \mathbb{N}$, which tends uniformly to some ρ. Because $\mathcal{R}_T$ is closed we know that $\rho \in \mathcal{R}_T$, hence if we establish the pointwise convergence of $P_{S,T}\rho_n$ towards $P_{S,T}\rho$, i.e.,

$$P_{S,T}\rho_n(s,t) \to P_{S,T}\rho(s,t) \qquad \text{for any} \quad (s,t) \in D_T, \tag{14}$$ {pointconv}

we can infer that $\rho(s,t) \leq P_{S,T}\rho(s,t)$ for any $(s,t) \in D_T$, i.e., $\rho \in \mathcal{R}_{S,T}$. Note that due to the hysteresis phenomenon, $P_{S,T}$ is not continuous in general, think of the case where ρ_n is such that f_{S,ρ_n} reaches a minimum value $1 + 1/n$ over – at least – a fixed ball inside D_T. Nevertheless we claim that

$$\tau_n(s) := \tau(S, \rho_n, s) \to \tau(s) := \tau(S, \rho, s) \qquad \text{for any } s \in [0, T), \tag{15}$$ {limtau}

and this property will suffice to establish (14). In order to prove the claim, denote

$$\tau^-(s) := \liminf_{n\to\infty} \tau_n(s) \quad \text{ and } \quad \tau^+(s) := \limsup_{n\to\infty} \tau_n(s),$$

which always satisfy $s \le \tau^-(s) \le \tau^+(s) \le T$, and first assume $\tau(s) < \tau^+(s)$: this would yield $\tau(s) < T$, hence $f_{S,\rho}(s, \tau(s)) = 1$, and by using the continuity of $\Phi_{s,t} : \mathcal{R}_T \to \mathbb{R},\ \rho \to f_{S,\rho}(s, t)$ we would have

{f1}
$$f_{S,\rho_n}(s, \tau(s)) \to f_{S,\rho}(s, \tau(s)) = 1. \tag{16}$$

Moreover there would be a subsequence $\rho_{\varphi(n)}$, $n \in \mathbb{N}$, and some $\alpha > 0$, such that

{tau+a}
$$\tau(s) + \alpha < \tau_{\varphi(n)}(s), \qquad \text{for } \ n \in \mathbb{N}, \tag{17}$$

and by using again the continuity of $\Phi_{s,t}$ this would give

$$f_{S,\rho}(s, t) = \lim_{n\to\infty} f_{S,\rho_{\varphi(n)}}(s, t) \ge 1, \qquad \text{for } \ t \le \tau(S, \rho, s) + \alpha,$$

which together with (16) implies that

{dtf}
$$0 \le \partial_t f_{S,\rho}(s, \tau(s)) = w\Big(S(\tau(s)) - \exp\big(\frac{\nu}{\rho(s, \tau(s))}\big)\Big). \tag{18}$$

Now, (17) would also yield $P_{S,T}\rho_{\varphi(n)}(s, \tau(s)) = f_{S,\rho_{\varphi(n)}}(s, \tau(s))$ for any n, according to the definition of $P_{S,T}$, hence

$$\begin{aligned} 1 \le \rho(s, \tau(s)) &= \lim_{n\to\infty} \rho_{\varphi(n)}(s, \tau(s)) \\ &\le \lim_{n\to\infty} P_{S,T}\rho_{\varphi(n)}(s, \tau(s)) \\ &= \lim_{n\to\infty} f_{S,\rho_{\varphi(n)}}(s, \tau(s)) = 1 \end{aligned}$$

where we have also used that $\rho \in \mathcal{R}_T$, $\rho_{\varphi(n)} \in \mathcal{R}_{S,T}$ for any n, and (16). In particular, $\rho(s, \tau(s)) = 1$, which contradicts (18): indeed any $S \in \mathcal{S}_T$ must satisfy $S(t) \le S_0 < e^\nu$. It follows that $\tau^+(s) \le \tau(s)$. Let us now assume $\tau^-(s) < \tau(s)$, and let $\rho_{\psi(n)}$, $n \in \mathbb{N}$, be a subsequence of ρ_n that satisfies

{tau-}
$$\tau_{\psi(n)}(s) < \tau(s) \qquad \text{and} \qquad \tau_{\psi(n)}(s) \to \tau^-(s). \tag{19}$$

Because this first imply $\tau_{\psi(n)}(s) < T$, we would have $f_{S,\rho_{\psi(n)}}(s, \tau_{\psi(n)}(s)) = 1$ for any n, hence

$$f_{S,\rho}(s, \tau^-(s)) = \lim_{n\to\infty} f_{S,\rho_{\psi(n)}}(s, \tau_{\psi(n)}(s)) = 1$$

by using the continuity of $\Phi_s : [s, T) \times \mathcal{R}_T \to \mathbb{R},\ (t, \rho) \to f_{S,\rho}(s, t)$. Now, the latter equality is clearly contradictory with $\tau^-(s) < \tau(s)$, therefore we must have $\tau^-(s) = \tau(s) = \tau^+(s)$, which proves our claim (15). We are then ready to prove the announced pointwise convergence property (14), and to do so we consider three cases, as follows:

(i) if $t < \tau(s)$, then from (15) we have $t < \tau_n(s)$ for n sufficiently large, hence

$$\lim_{n\to\infty} P_{S,T}\rho_n(s, t) = \lim_{n\to\infty} f_{S,\rho_n}(s, t) = f_{S,\rho}(s, t) = P_{S,T}\rho(s, t),$$

by using again the continuity of the mapping $\Phi_{s,t} : \rho \to f_{S,\rho}(s, t)$.

(ii) if $\tau(s) < t$, then we have $\tau_n(s) < t$ for n sufficiently large, and

$$\lim_{n\to\infty} P_{S,T}\rho_n(s, t) = 1 = P_{S,T}\rho(s, t).$$

(iii) finally if $\tau(s) = t < T$, then $P_{S,T}\rho(s,t) = f_{S,\rho}(s,t) = 1$, and denoting

$$\mathbb{N}_- := \{n \in \mathbb{N} : \tau_n(s) < \tau(s) = t\}, \qquad \mathbb{N}_+ := \mathbb{N} \setminus \mathbb{N}_-,$$

we observe that $P_{S,T}\rho_n(s,t) = 1$ for $n \in \mathbb{N}_-$, whereas

$$\lim_{n\to\infty} P_{S,T}\rho_n(s,t) = \lim_{n\to\infty} f_{S,\rho_n}(s,t) = f_{S,\rho}(s,t) = 1 \qquad \text{for } n \in \mathbb{N}_+.$$

At this point we have shown that $\mathcal{R}_{S,T}$ is a *closed, non-empty* subset of $\mathcal{R}_T$ (equipped with the sup norm). ⋄

Let us now prove the embedding (11), i.e., that $P_{S,T}$ *maps* $\mathcal{R}_{S,T}$ *into itself.* For this purpose we consider an arbitrary $\rho \in \mathcal{R}_{S,T}$. Since we have $P_{S,T}\rho \geq 1$ by construction, and

$$\rho \leq P_{S,T}\rho \qquad \Longrightarrow \qquad P_{S,T}\rho \leq P_{S,T}(P_{S,T}\rho)$$

by using Lemma 2.3, we only need to show that $P_{S,T}\rho$ is continuous, and more precisely, that it is continuous with respect to its first argument s since it is clearly Lipschitz with respect to t, see (10). Let us then take $s, s', t \in [0,T)$ such that $s < s' \leq t$, and estimate the difference $\Delta(s,s',t) := P_{S,T}\rho(s',t) - P_{S,T}\rho(s,t)$ by distinguishing the relative positions of t, $\tau(s) := \tau(S,\rho,s)$ and $\tau(s') := \tau(S,\rho,s')$. We have

$$\Delta(s,s',t) = 0 \qquad \text{if } \max\{\tau(s),\tau(s')\} < t,$$

and

$$\Delta(s,s',t) = \underbrace{\frac{\nu}{\ln S(s')} - \frac{\nu}{\ln S(s)}}_{=:\ \delta_1(s,s',t)} + \underbrace{w\int_s^{s'} e^{\frac{\nu}{\rho(s,t'}} - S(t')\,\mathrm{d}t'}_{=:\ \delta_2(s,s',t)} + \underbrace{w\int_{s'}^{t} e^{\frac{\nu}{\rho(s,t'}} - e^{\frac{\nu}{\rho(s',t'}}\,\mathrm{d}t'}_{=:\ \delta_3(s,s',t)}$$

if $t \geq \min\{\tau(s),\tau(s')\}$. For the intermediate cases we can check that

$$\big(\Delta - (\delta_1+\delta_2+\delta_3)\big)(s,s',t) = \begin{cases} w\int_{\tau(s)}^t S(t') - e^{\frac{\nu}{\rho(s,t'}}\,\mathrm{d}t' & \text{if } \tau(s) \leq t < \tau(s') \\ -w\int_{\tau(s')}^t S(t') - e^{\frac{\nu}{\rho(s',t'}}\,\mathrm{d}t' & \text{if } \tau(s') \leq t < \tau(s). \end{cases}$$

Now if $\tau(s) < \tau(s')$ (the other case will follow by the same argument), because $\tau(s) < T$ we observe that

$$P_{S,T}\rho(s',t') \leq 1 = P_{S,T}\rho(s,t') \qquad \text{for } t' \geq \tau(s) \tag{20}$$
{ss'}

which first implies that $\Delta(s,s',t) \geq 0$, and second by using $1 \leq \rho \leq P_{S,T}\rho$, that $\rho(s,t') = 1$ for $t' \geq \tau(s)$, hence

$$w\int_{\tau(s)}^t S(t') - e^{\frac{\nu}{\rho(s,t'}}\,\mathrm{d}t' = w\int_{\tau(s)}^t S(t') - e^{\nu}\,\mathrm{d}t' \leq 0 \qquad \text{for } t' \geq \tau(s).$$

Therefore we have

$$0 \leq \Delta(s,s',t) \leq (\delta_1+\delta_2+\delta_3)(s,s',t)$$

in the case where $\tau(s) \leq t < \tau(s')$, and the same reasoning yields the reverse inequalities in the case where $\tau(s') \leq t < \tau(s)$. Summing up, we see that we always have

$$|\Delta(s,s',t)| \leq \big|(\delta_1+\delta_2+\delta_3)(s,s',t)\big|,$$

and we easily check that every $\delta_i(s,s',t)$ goes to 0 as $s \to s'$, which establishes the continuity of $P_{S,T}\rho$, hence the embedding (11). ⋄

In order to finish the proof it remains to show that the operator $P_{S,T}$ is *contracting* over $(\mathcal{R}_{S,T}, \|\cdot\|_{\infty,k})$. To do so, let us take ρ_1, ρ_2 in $\mathcal{R}_{S,T}$ and estimate the difference $D(s,t) := P_{S,T}\rho_2(s,t) - P_{S,T}\rho_1(s,t)$ according to the relative positions of t, $\tau_1(s) := \tau(S,\rho_1,s)$ and $\tau_2(s) := \tau(S,\rho_2,s)$. Clearly, we have $D(s,t) = 0$ for $t \geq \max\{\tau_1(s), \tau_2(s)\}$, and

$$D(s,t) = w\int_s^t \exp\Big(\frac{\nu}{\rho_1(s,t')}\Big) - \exp\Big(\frac{\nu}{\rho_2(s,t')}\Big)\,\mathrm{d}t' =: d(s,t)$$

for $t \leq \min\{\tau_1(s), \tau_2(s)\}$. In the intermediate case where $\tau_1(s) \leq t < \tau_2(s)$, we have

$$D(s,t) = d(s,t) + w\int_{\tau_1(s)}^t S(t') - \exp\Big(\frac{\nu}{\rho_1(s,t')}\Big)\,\mathrm{d}t'.$$

Now, by observing that

$$1 \leq \rho_1(s,t) \leq P_{S,T}\rho_1(s,t) = 1 \leq P_{S,T}\rho_2(s,t) \qquad \text{for } t > \tau_1(s)$$

we first see that $D(s,t) \geq 0$ in this case. Second we infer $\rho_1(s,t) = 1$ for $t > \tau_1(s)$, hence according to $S(t') \leq S_0 < e^\nu$, $\int_{\tau_1(s)}^t S(t') - \exp\big(\frac{\nu}{\rho_1(s,t')}\big)\,\mathrm{d}t' \leq 0$, which finally yields $0 \leq D(s,t) \leq d(s,t)$. Note that by symmetry, we have $0 \geq D(s,t) \geq d(s,t)$ when $\tau_2(s) \leq t < \tau_1(s)$. Summing up, we find that $|D(s,t)| \leq |d(s,t)|$ for any $(s,t) \in D_T$, hence (remember that $k = w\nu e^\nu$)

$$|D(s,t)| \leq k\int_s^t |\rho_2(s,t') - \rho_1(s,t')|\,\mathrm{d}t'$$

indeed the mapping $x \in [1,\infty) \to e^{\nu/x}$ has Lipschitz constant νe^ν. Next we compute

$$\begin{aligned}
k\int_s^t |\rho_2(s,t') - \rho_1(s,t')|\,\mathrm{d}t' &\leq \int_s^t ke^{k(t'-s)}\,\mathrm{d}t'\|\rho_2 - \rho_1\|_{\infty,k} \\
&\leq (e^{k(t-s)} - 1)\|\rho_2 - \rho_1\|_{\infty,k} \\
&\leq e^{k(t-s)}L_k\|\rho_2 - \rho_1\|_{\infty,k}
\end{aligned}$$

with $L_k := 1 - e^{-kT} < 1$, and it follows that

$$\|P_{S,T}\rho_2 - P_{S,T}\rho_1\|_{\infty,k} := \sup_{(s,t)\in D_T} e^{-k(t-s)}|D(s,t)| \leq L_k\|\rho_2 - \rho_1\|_{\infty,k},$$

which shows that $P_{S,T}$ is indeed *contracting* on $(\mathcal{R}_{S,T}, \|\cdot\|_{\infty,k})$, and ends the proof of Lemma 2.1. $\square$

3. **Continuous solutions to the complete system.** In this section we shall establish the existence and uniqueness of solutions to the complete system (6) with the aid of the operator $Q_T : \mathcal{R}_T \to \mathcal{C}([0,T);\mathbb{R})$, defined for any $T < \infty$ by (8), i.e.,

$$Q_TS(t) := S_0 - J\int_0^t \exp\Big(-\frac{\nu\rho_S^2(s,s)}{2}\Big)\big(\rho_S^3(s,t) - 1\big)\,\mathrm{d}s$$

where ρ_S denotes the unique solution of the auxiliary system (7) – or equivalently, the unique fixed point of $P_{S,T}$ – which existence is guaranteed by Theorem 2.2. Indeed, it is readily checked that the couple (S,ρ_S) is a solution of (6) (on $\mathcal{S}_T \times \mathcal{R}_T$) if and only if S is a fixed point of Q_T.

3.1. **Existence of maximal solutions.** Again we shall apply the Picard-Banach fixed point theorem, and in order to do so we first establish the following result.

Lemma 3.1. *Let $\alpha > 1$. For any S, Z in the space*

$$\mathcal{S}_{T,\alpha} := \{S \in \mathcal{S}_T : S(t) \geq \alpha,\ t \in [0,T)\},$$

and any $t \in [0,T)$, we have

$$|Q_T Z(t) - Q_T S(t)| \leq C_{T,\alpha} \int_0^t |Z(s) - S(s)|\,\mathrm{d}s \tag{21}$$ {Qlip}

for a constant $C_{T,\alpha}$ that depends on T, α, ν, w and J.

For later purposes we let

$$q_S(s,t) := \exp\Big(-\frac{\nu\rho_S^2(s,s)}{2}\Big)\big(\rho_S^3(s,t) - 1\big) \tag{22}$$ {qS}

so that $Q_T S(t) = S_0 - J\int_0^t q_S(s,t)\,\mathrm{d}s$, and denote by $C_{T,\alpha}$ a generic constant that depends on T and α (in addition to ν, w, J), and which value may vary at each occurrence.

Proof. Let us start with a uniform bound for ρ: for any $S \in \mathcal{S}_{T,\alpha}$ we have

$$|\rho_S(s,s)| = \Big|\frac{\nu}{\ln S(s)}\Big| \leq \frac{\nu}{\ln\alpha} \quad \text{for all } s \in [0,T)$$

hence out of the diagonal we find, according to (13),

$$|\rho_S(s,t)| \leq |\rho_S(s,s)| + w(t-s)(e^\nu - 1) \leq \frac{\nu}{\ln\alpha} + wT(e^\nu - 1). \tag{23}$$ {boundrho}

Turning to the difference $\Delta(s,t) := \rho_Z(s,t) - \rho_S(s,t)$, we first check that

$$|\Delta(s,s)| = \frac{\nu}{\ln^2\alpha}|Z(s) - S(s)| \quad \text{for all } s \in [0,T).$$

and in order to write estimates out of the diagonal, we distinguish between different cases corresponding to the relative positions of t, $\tau_S(s) := \tau(S,\rho_S,s)$ and $\tau_Z(s) := \tau(Z,\rho_Z,s)$: for $t \geq \max\{\tau_S(s),\tau_Z(s)\}$, we clearly have $\Delta(s,t) = 0$, and

$$\Delta(s,t) = \Delta(s,s) + \underbrace{w\int_s^t (Z-S)(t') - \big(e^{\frac{\nu}{\rho_Z(s,t')}} - e^{\frac{\nu}{\rho_S(s,t')}}\big)\,\mathrm{d}t'}_{=:\ \delta(s,t)}$$

for $t \leq \min\{\tau_S(s),\tau_Z(s)\}$. Now, in the intermediate case $\tau_S(s) \leq t < \tau_Z(s)$, we find (by using that $\tau_S(s) < T$)

$$\rho_Z(s,t') \geq 1 = P_{S,T}\rho_S(s,t') = \rho_S(s,t') \quad \text{for } t' \geq \tau_S(s),$$

hence

$$0 \leq \Delta(s,t) = \Delta(s,s) + \delta(s,t) + w\int_{\tau_S(s)}^t S(t') - e^{\frac{\nu}{\rho_S(s,t')}}\,\mathrm{d}t' \leq \Delta(s,s) + \delta(s,t).$$

Note that by symmetry, $0 \geq \Delta(s,t) \geq \Delta(s,s) + \delta(s,t)$ when $\tau_Z(s) \leq t < \tau_S(s)$. Therefore we have for any $(s,t) \in D_T$

$$\begin{aligned}
|\Delta(s,t)| &\leq |\Delta(s,s)| + |\delta(s,t)| \\
&\leq C_{T,\alpha}\Big(|Z(s) - S(s)| + \int_s^t |Z(t') - S(t')| + |\Delta(s,t')| \,\mathrm{d}t'\Big) \\
&\leq C_{T,\alpha}\Big(|Z(s) - S(s)| + \int_s^t |Z(t') - S(t')| \,\mathrm{d}t'\Big)
\end{aligned}$$

where we have used (again) that $x \in [1,\infty) \to e^{\nu/x}$ has Lipschitz constant νe^{ν} in the second inequality, and the Gronwall Lemma in the third inequality. As for the differences $q_Z - q_S$, we find by using (22) and (23)

$$|q_Z(s,t) - q_S(s,t)| \leq C_{T,\alpha}\big(|\Delta(s,s)| + |\Delta(s,t)|\big),$$

hence

$$\begin{aligned}
|Q_T Z(t) - Q_T S(t)| &\leq \int_0^t |q_Z(s,t) - q_S(s,t)| \,\mathrm{d}s \\
&\leq C_{T,\alpha} \int_0^t |\Delta(s,s)| + |\Delta(s,t)| \,\mathrm{d}s \\
&\leq C_{T,\alpha} \int_0^t \Big(|Z(s) - S(s)| + \int_s^t |Z(t') - S(t')| \,\mathrm{d}t'\Big) \,\mathrm{d}s \\
&\leq C_{T,\alpha} \int_0^t |Z(s) - S(s)| \,\mathrm{d}s
\end{aligned}$$

which ends the proof. □

This yields the following result, which we shall next extend to global solutions.

Theorem 3.2. *There exists a unique maximal solution to the complete system* (6). *More precisely, for any S_0 there is a time $T > 0$ such that Q_T possess a unique fixed point (which provides a solution of* (6) *on $\mathcal{S}_T \times \mathcal{R}_T$), moreover if S_1 and S_2 are the respective fixed points of Q_{T_1} and Q_{T_2} with $T_1 < T_2$, then $S_1 = S_2|_{[0,T_1)}$.*

Proof. Let S_0 be fixed, and consider two positive times T and T^* that satisfy $0 < T \leq T^* < \infty$. According to (13), we have $|\rho_S(s,t)| \leq |\rho_S(s,s)| + wT^*(e^{\nu} - 1)$ on D_T, hence

{qbound} $$q_S(s,t) \leq \exp\Big(-\frac{\nu\rho_S^2(s,s)}{2}\Big)\big((|\rho_S(s,s)| + wT^*(e^{\nu} - 1))^3 + 1\big) \leq C_* \tag{24}$$

with a constant that only depends on T^* and on the dimensionless parameters w, ν. By chosing $T \leq \min\{T^*, \frac{S_0}{2JC_*}\}$, we then get

$$Q_T S(t) = S_0 - J\int_0^t q_S(s,t) \,\mathrm{d}s \geq S_0 - JTC_* \geq S_0/2,$$

for any $S \in \mathcal{S}_T$, in particular we see that Q_T maps the Banach space $(\mathcal{S}_{T,S_0/2}, L^{\infty})$ into itself. In order to apply the Picard-Banach theorem (more precisely, its p-th iterate corollary) we next estimate $n_p(t) := |(Q_T)^p Z(t) - (Q_T)^p S(t)|$ for Z, S in $\mathcal{S}_{T,S_0/2}$ with a classical bootstrap technique: since $n_0(s) \leq \|Z - S\|_{L^{\infty}(0,T)}$ for any

$s \in [0,T)$ and according to Lemma 3.1, we have by induction for any $p \geq 1$

$$n_p(t) \leq C_0 \int_0^t n_{p-1}(s)\,\mathrm{d}s \leq C_0 \int_0^t \left(\frac{(C_0 s)^{p-1}}{(p-1)!} \|Z-S\|_{L^\infty(0,T)} \right) \mathrm{d}s \leq \frac{(C_0 t)^p}{p!} \|Z-S\|_{L^\infty(0,T)}$$

where $C_0 := C_{T,S_0/2}$ denotes the (fixed) constant appearing in (21). It follows that for sufficiently large p the iterate $(Q_T)^p$ is a contractive mapping on $(\mathcal{S}_{T,S_0/2}, L^\infty)$, and by the Picard-Banach theorem this yields the existence of a unique fixed point $S \in \mathcal{S}_{T,S_0/2}$ of Q_T, hence a unique solution $(\rho, S) = (\rho_S, S)$ to the system (6) corresponding to the final time T. In order to show that two solutions S_1, S_2 corresponding to different times $T_1 < T_2$ coincide on $[0,T_1)$, let us observe that if $S_1^* := S_2|_{[0,T_1)}$, the associated partial solution ρ_1^* (defined as the unique fixed point of $P_{S_1^*,T_1}$) coincides with ρ_2 on D_{T_1}, hence $q_{S_1^*}$ coincides with q_{S_2} on $[0,T_1)$, and in particular we have

$$Q_{T_1} S_1^*(t) = S_0 - J \int_0^t q_{S_1^*}(s)\,\mathrm{d}s = S_0 - J \int_0^t q_{S_2}(s)\,\mathrm{d}s = S_2(t) = S_1^*(t) \quad \text{for} \quad t < T_1,$$

i.e., $S_1^*(t)$ is a fixed point of Q_{T_1}. The announced result clearly follows from the uniqueness of this fixed point. ☐

Note that we just have seen that if the function $S_2 \in \mathcal{S}_{T_2}$ is a fixed point of Q_{T_2}, then its restriction to any interval $[0,T_1)$ with $T_1 < T_2$ is a fixed point of Q_{T_1}. In particular, for all initial data S_0 there exists a largest time

$$T_{\max} := \sup\{T > 0 : Q_T \text{ admits a fixed point in } \mathcal{S}_T\} \tag{25}$$ {Tmax}

for the maximal solution to (6). According to Theorem 3.2 we already know that this time is always positive. The remainder of this section is devoted to show that it can never be finite.

3.2. Existence of global solutions.

Let us begin with an intermediate result.

Lemma 3.3. *If the maximal time $T_{\max}$ given by (25) is finite, then the maximal solution (ρ, S) to (6) satisfies*

$$S(t) \to 1 \quad \text{as} \quad t \to T_{\max}.$$

Proof. Since S (restricted to $[0,T)$) is a fixed point of Q_T for any $T < T_{\max}$, we have

$$S'(t) = -J \left(\int_0^t \partial_t q_S(s,t)\,\mathrm{d}s + q_S(t,t) \right) \quad \text{for} \quad t < T_{\max} \tag{26}$$ {Sder}

with $q_S(s,t) = \exp\big(-\frac{\nu\rho^2(s,s)}{2}\big)(\rho^3(s,t)-1) \geq 0$. In particular, we know from (24) that q_S is bounded on any bounded set, and by using (13) and (23) we find that

$$|\partial_t q_S(s,t)| = 3\exp\Big(-\frac{\nu\rho^2(s,s)}{2}\Big)\rho^2(s,t)|\partial_t\rho(s,t)| \leq C\exp\Big(-\frac{\nu\rho^2(s,s)}{2}\Big)(\rho^2(s,s)+1) \leq C' \tag{27}$$ {qlip t}

with constants C, C' depending on $T_{\max}$, w and ν, hence S' is bounded on the bounded interval $[0,T_{\max})$. In particular, it is easy to check that S possess a limit on $T_{\max}$, which we shall denote by $S(T_{\max})$ and which, by construction, is larger or equal to 1. Let us now show that if $S(T_{\max}) > 1$, it would be possible to extend the maximal solution beyond $T_{\max}$, yielding a contradiction. To do so we first consider an auxiliary time $\bar{T} \in (T_{\max}, 2T_{\max}]$ and introduce the set

$$\mathcal{S}_{T_{\max},\bar{T}} := \{Z \in \mathcal{C}([T_{\max}, \bar{T})) : Z(T_{\max}) = S(T_{\max}) \;\text{ and }\; \alpha \le Z \le S_0\}$$

with $\alpha := (S(T_{\max}) - 1)/2 > 1$, then we construct a new operator $Q_{T_{\max},\bar{T}}$ on $\mathcal{S}_{T_{\max},\bar{T}}$ as follows. For any $Z \in \mathcal{S}_{T_{\max},\bar{T}}$ we denote by $\bar{Z}$ its continuous extension to $[0, \bar{T})$ obtained by stitching it to S, i.e.,

$$\bar{Z}(t) := S(t)\mathbf{1}_{[0,T_{\max})}(t) + Z(t)\mathbf{1}_{[T_{\max},\bar{T})}(t) \quad \text{for} \quad t \in [0, \bar{T}),$$

and then we set

$$Q_{T_{\max},\bar{T}}Z(t) := S(T_{\max}) - J\int_0^{T_{\max}} \big(q_{\bar{Z}}(s,t) - q_{\bar{Z}}(s,T_{\max})\big)\,\mathrm{d}s - J\int_{T_{\max}}^{t} q_{\bar{Z}}(s,t)\,\mathrm{d}s$$

for any t in $[T_{\max}, \bar{T})$. Let us observe that up to choosing $\bar{T}$ close enough to $T_{\max}$, the resulting $Q_{T_{\max},\bar{T}}Z$ is bounded below by α: indeed, according to (24) and (27), there exists constants c, c' depending only on $T_{\max}, \nu, w$ and J such that

$$\begin{aligned} Q_{T_{\max},\bar{T}}Z(t) &\ge S(T_{\max}) - J\bar{T}\sup_{s\le T_{\max}} |q_{\bar{Z}}(s,t) - q_{\bar{Z}}(s,T_{\max})| - J|t - T_{\max}|c \\ &\ge S(T_{\max}) - |t - T_{\max}|c'. \end{aligned}$$

Hence by choosing $\bar{T}$ so that

$$T_{\max} < \bar{T} \le T_{\max} + \min\{T_{\max}, (S(T_{\max}) - \alpha)/c'\},$$

we find that $Q_{T_{\max},\bar{T}}Z(t) \ge \alpha$ for all $t \in [T_{\max}, \bar{T})$. As in the proof of Theorem 3.2, we next observe that because S and $\bar{Z}$ coincide on $[0, T_{\max})$, q_S and $q_{\bar{Z}}$ coincide on $D_{T_{\max}}$. In particular, we have

{Qtbis}
$$Q_{T_{\max},\bar{T}}Z(t) = \underbrace{S(T_{\max}) + J\int_0^{T_{\max}} q_S(s,T_{\max})\,\mathrm{d}s}_{S_0} - J\int_0^t q_{\bar{Z}}(s,t)\,\mathrm{d}s, \tag{28}$$

which shows that $Q_{T_{\max},\bar{T}}Z$ is nothing but the restriction of $Q_{\bar{T}}\bar{Z}$ to $[T_{\max}, \bar{T})$. From these facts we can infer that $Q_{\bar{T}}$ maps the Banach set

$$\mathcal{S}^S_{\bar{T}} := \{Z \in \mathcal{C}([0, \bar{T})) : Z|_{[0,T_{\max})} = S \;\text{ and }\; Z|_{[T_{\max},\bar{T})} \in \mathcal{S}_{T_{\max},\bar{T}}\}$$

into itself. Because this is a subset of $\mathcal{S}_{\bar{T}}$, the arguments detailed in the proof of Theorem 3.2 show that one iterate of $Q_{\bar{T}}$ is contractive on $\mathcal{S}^S_{\bar{T}}$. Therefore $Q_{\bar{T}}$ possess one fix point in the small set, hence in $\mathcal{S}_{\bar{T}}$, which yields a contradiction with the definition of $T_{\max}$. □

According to the above lemma, we now know that the maximal solution S is continuous on the closed interval $[0, T_{\max}]$ in the case where the latter is bounded. In order to prove our Main theorem, we are thus left to show the following

Lemma 3.4. *The maximal time $T_{\max}$ given by (25) is infinite.*

Proof. In order to establish this result we will assume that $T_{\max}$ is finite, and show that Lemma 3.3 yields a contradiction. First, observe that since ρ is Lipschitz with respect to t, it is continuous on every closed interval $[s, T_{\max}]$ with $s < T_{\max}$. Next we introduce the set

$$A := \{s \in [0, T_{\max}) : \tau(S, \rho, s) = T_{\max}\}$$

and remember from Section 2 that $\partial_t\rho(s,T_{\max}) = 0$ holds for all $s \notin A$, whereas for $s \in A$ we have $\partial_t\rho(s,t) = w(S(t) - \exp\big(\frac{\nu}{\rho(s,t)}\big))$. According to the above observation, the latter expression is continuous on the closed interval $[s,T_{\max}]$, hence by using Lemma 3.3 we have

$$\partial_t\rho(s,T_{\max}) = w\left(1 - \exp\Big(\frac{\nu}{\rho(s,T_{\max})}\Big)\right) < 0 \quad \text{for all} \quad s \in A,$$

since $\rho \geq 1$ by construction. Now, assume for a moment that the set A is of zero measure: since $\rho(s,T_{\max}) = 1$ for any s outside A, this would yield

$$S(T_{\max}) = S_0 - J\int_A \exp\Big(-\frac{\nu\rho^2(s,s)}{2}\Big)(\rho^3(s,T_{\max}) - 1)\,\mathrm{d}s = S_0 > 1,$$

hence a contradiction with Lemma 3.3. It follows that A is of positive measure. Let us now consider the derivative of S. By using again Lemma 3.3 we observe that $\rho(t,t) \to \infty$ as $t \to T_{\max}$ hence

$$q_S(t,t) = \exp\Big(-\frac{\nu\rho^2(t,t)}{2}\Big)\big(\rho^3(t,t) - 1\big) \to 0 \quad \text{as} \quad t \to T_{\max}.$$

In addition we note that for all $s \in A$,

$$\lim_{t\to T_{\max}} \partial_t q_S(s,t) = 3\exp\Big(-\frac{\nu\rho^2(s,s)}{2}\Big)\rho^2(s,T_{\max})\partial_t\rho(s,T_{\max}) < 0,$$

whereas this limit vanishes for $s \notin A$. By gathering the above arguments we find

$$\lim_{t\to T_{\max}} S'(t) = -J\int_0^{T_{\max}} \partial_t q_S(s,T_{\max})\,\mathrm{d}s = -J\int_A \partial_t q_S(s,T_{\max})\,\mathrm{d}s > 0,$$

which clearly contradicts the fact that $S(t) > 1 = S(T_{\max})$ for $t < T_{\max}$. □

4. **Stability with respect to the parameters.** In this section we shall establish the following stability result, which completes our Main Theorem.

Theorem 4.1. *If (ρ,S) and $(\tilde\rho,\tilde S)$ are the solutions (in $\mathcal{R}_\infty \times \mathcal{S}_\infty$) to 6 corresponding to admissible sets of data $D := (\nu,w,J,S_0)$ and $\tilde D := (\tilde\nu,\tilde w,\tilde J,\tilde S_0)$, respectively, then for any $T < \infty$ we have*

$$\|S - \tilde S\|_{L^\infty(0,T)} \leq C\|D - \tilde D\|_{\ell^\infty}$$

for a constant C that only depends on T and on $M := \|(D,\tilde D)\|_{\ell^\infty}$.

Remark 1. As already pointed out in the introduction, we shall emphasize that no stability is expected to hold with respect to ρ, as this variable is not even bounded on compact sets.

Proof. For sake of conciseness we write in this proof $\Delta_S := S - \tilde S$, $\Delta_\rho := \rho - \tilde\rho$, $\Delta_w := w - \tilde w$, ... and finally $\Delta_D := D - \tilde D$ (for simplicity we shall write $|\Delta_D|$ instead of $\|\Delta_D\|_{\ell^\infty}$). Moreover by C we will denote a generic constant depending on M and T, and which value may vary at each occurrence. Let us now introduce

$$\varepsilon(s) := \exp\Big(-\frac{\nu}{6}\rho^2(s,s)\Big) = \exp\Big(-\frac{\nu^3}{6}\ln^{-2}\big(S(s)\big)\Big) \quad \text{and} \quad r(s,t) := \varepsilon(s)\rho(s,t),$$

so that $q(s,t) = q_S(s,t) = r^3(s,t) - \varepsilon^3(s)$, and hence

$$S(t) = S_0 - J\int_0^t \big(r^3(s,t) - \varepsilon^3(s)\big)\,\mathrm{d}s.$$

As it can be easily checked, $\varepsilon(s)$ and $r(s,t)$ are bounded: indeed we clearly have $0 < \varepsilon(s) \leq 1$, and it follows from (23) that

$$0 \leq r(s,t) \leq \varepsilon(s)\big(\rho(s,s) + C\big) \leq C. \tag{29}$$ {boundr}

As a first consequence we see that

{DS} $$|\Delta_S(t)| \leq C\Big(|\Delta_D| + \int_0^t |\Delta_r(s,t)| + |\Delta_\varepsilon(s)| \,\mathrm{d}s\Big). \tag{30}$$

It then remains to bound Δ_r and Δ_ε in terms of Δ_S (via the equation satisfied by ρ), and apply a gronwall Lemma to conclude. Let us carry out this program: by observing that both the functions $\phi : x \in (1, M] \to \exp\big(-\nu^3 \ln^{-2}(x)/6\big)$ and $\psi : x \in (1, M] \to \exp\big(-\nu^3 \ln^{-2}(x)/6\big)\nu \ln^{-1}(x)$ are uniformly Lipschitz, we first find that

{inegve} $$\max\{|\Delta_\varepsilon(s)|, |\Delta_r(s,s)|\} \leq C(|\Delta_\nu| + |\Delta_S(s)|). \tag{31}$$

Next, we turn to Δ_ρ and remind that according to Section 2, ρ reads

$$\rho(s,t) = \begin{cases} \rho(s,s) + w \int_s^t S(t') - e^{\frac{\nu}{\rho(s,t')}} \,\mathrm{d}t' & \text{if } s \leq t \leq \tau(s) := \tau(S,\rho,s) \\ 1 & \text{if } \tau(s) < t \end{cases}$$

(and similarly for $\tilde\rho$, with $\tau(\tilde S, \tilde\rho, s)$ denoted $\tilde\tau(s)$ for conciseness). It follows that for any $(s,t) \in D_T$ such that $t > \max\{\tau(s), \tilde\tau(s)\}$, we have $|\Delta_\rho(s,t)| = 0$, whereas for $s \leq t \leq \min\{\tau(s), \tilde\tau(s)\}$, it holds

{inrho} $$|\Delta_\rho(s,t)| \leq |\Delta_\rho(s,s)| + \int_s^t |wS(t') - \tilde w \tilde S(t')| \,\mathrm{d}t' + \int_s^t |we^{\frac{\nu}{\rho(s,t')}} - \tilde w e^{\frac{\tilde\nu}{\tilde\rho(s,t')}}| \,\mathrm{d}t'. \tag{32}$$

As for the remaining cases, say for $\tilde\tau(s) < t \leq \tau(s)$, we have $\tilde\rho(s,t') = 1$ for $t' > \tilde\tau(s)$ and $\tilde S < e^{\tilde\nu}$, hence

$$\begin{aligned} 0 \leq \Delta_\rho(s,t) \leq \Delta_\rho(s,s) + \int_s^t \Big(wS(t') - \tilde w \tilde S(t')\Big) \,\mathrm{d}t' + \int_s^t \Big(\tilde w e^{\frac{\tilde\nu}{\tilde\rho(s,t')}} - we^{\frac{\nu}{\rho(s,t')}}\Big) \,\mathrm{d}t' \\ + \tilde w \int_{\tilde\tau(s)}^t \Big(\underbrace{\tilde S(s) - e^{\frac{\tilde\nu}{\tilde\rho(s,t')}}}_{\leq 0}\Big) \,\mathrm{d}t' \end{aligned}$$

so that (32) holds for all $(s,t) \in D_T$.

Let us now estimate the two last terms in the corresponding right hand side. With straightforward computations (and by making use of the intrinsic bounds satisfied by the solutions), we find that

{inrho1} $$|wS(t') - \tilde w \tilde S(t')| \leq C\big(|\Delta_w| + |\Delta_S(t')|\big) \tag{33}$$

and

{inrho2} $$\begin{aligned} |we^{\frac{\nu}{\rho(s,t')}} - \tilde w e^{\frac{\tilde\nu}{\tilde\rho(s,t')}}| &\leq C\Big(|\Delta_w| + |\Delta_\nu| \Big|e^{\frac{\nu}{\rho(s,t')}} - e^{\frac{\nu}{\tilde\rho(s,t')}}\Big|\Big) \\ &\leq C\Big(|\Delta_w| + |\Delta_\nu| + |\Delta_\rho(s,t')|\Big). \end{aligned} \tag{34}$$

By gathering (32), (33), (34) and applying the Gronwall lemma, we thus obtain

$$\begin{aligned}
|\Delta_\rho(s,t)| &\le |\Delta_\rho(s,s)| + C\int_s^t \Big(|\Delta_w| + |\Delta_S(t')| + |\Delta_\nu| + |\Delta_\rho(s,t')|\Big)\,\mathrm{d}t' \\
&\le |\Delta_\rho(s,s)| + C\int_s^t \Big(|\Delta_w| + |\Delta_S(t')| + |\Delta_\nu|\Big)\,\mathrm{d}t' \qquad (35)\\
&\le |\Delta_\rho(s,s)| + C\Big(|\Delta_D| + \int_s^t |\Delta_S(t')|\,\mathrm{d}t'\Big).
\end{aligned}$$

{Drho}

In order to finally write an estimate for Δ_r, we next observe that (on D_T)

$$\Delta_r = \varepsilon\rho - \tilde{\varepsilon}\tilde{\rho} = (\varepsilon\tilde{\varepsilon})^{1/2}\Delta_\rho + (\varepsilon^{1/2}\rho + \tilde{\varepsilon}^{1/2}\tilde{\rho})(\varepsilon^{1/2} - \tilde{\varepsilon}^{1/2}).$$

This is helpful indeed, since by using that ε and $\tilde{\varepsilon}$ are uniformly bounded (by 1), we can first check that, as in (31), we have

$$(\varepsilon\tilde{\varepsilon})^{1/2}(s)|\Delta_\rho(s,s)| \le |\varepsilon^{1/2}(s)\rho(s,s) - \tilde{\varepsilon}^{1/2}(s)\tilde{\rho}(s,s)| \le C\big(|\Delta_\nu| + |\Delta_S(s)|\big),$$

second, by using (23) and computing as in (29),

$$\varepsilon^{1/2}(s)|\rho(s,t)| \le \varepsilon^{1/2}(s)\big(|\rho(s,s)| + C\big) \le C$$

(and similarly for $\tilde{\varepsilon}^{1/2}\tilde{\rho}$), and third, again as in (31),

$$|\varepsilon^{1/2}(s) - \tilde{\varepsilon}^{1/2}(s)| \le C\big(|\Delta_\nu| + |\Delta_S(s)|\big).$$

According to (35), it thus follows that

$$\begin{aligned}
|\Delta_r(s,t)| \le (\varepsilon\tilde{\varepsilon})^{1/2}(s)\Big(|\Delta_\rho(s,s)| + C\big(|\Delta_D| + \int_s^t |\Delta_S(t')|\,\mathrm{d}t'\big)\Big) \\
+ \big(\varepsilon^{1/2}(s)\rho(s,t) + \tilde{\varepsilon}^{1/2}(s)\tilde{\rho}(s,t)\big)(\varepsilon^{1/2}(s) - \tilde{\varepsilon}^{1/2}(s)) \\
\le C\Big(|\Delta_D| + |\Delta_S(s)| + \int_s^t |\Delta_S(t')|\,\mathrm{d}t'\Big).
\end{aligned}$$

Plugging this and (31) into (30) yields then (for any $t \in [0,T)$)

$$\begin{aligned}
|\Delta_S(t)| \le C\Big(|\Delta_D| + \int_0^t |\Delta_S(s)| + \Big(\int_s^t |\Delta_S(t')|\,\mathrm{d}t'\Big)\,\mathrm{d}s\Big) \\
\le C\Big(|\Delta_D| + 2\int_0^t |\Delta_S(s)|\,\mathrm{d}s\Big) \le C|\Delta_D|
\end{aligned}$$

by using the Gronwall lemma, which ends this proof. □

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